

Holographic Shell Model: Stack Data Structure inside Black Holes

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We suggest that bits of information inhabit, universally and holographically, the entire black hole interior, a bit per a light sheet unit interval of order Planck area difference. The number of distinguishable (tagged by a binary code) configurations, counted within the context of a discrete holographic shell model, is given by the Catalan series. The area entropy formula is recovered, including the universal logarithmic correction, and the equipartition of mass per degree of freedom is proven. The black hole information storage resembles a stack data structure.

The 't Hooft-Susskind holographic principle[1] asserts that all of information contained in some region of space can be represented as a 'hologram' on the boundary of that region. It furthermore puts a universal purely geometrical bound, saturated by Bekenstein-Hawking area entropy formula [2], on the amount of entropy stored within that region, namely

$$S \leq \frac{A}{4G} . \quad (1)$$

Here, A denotes the area of the closed spacial boundary, G is Newton's constant, and $\hbar = c = k_B = 1$. Sticking momentarily to spherical symmetry, a covariant [3] local generalization of the holographic bound, that is

$$S(r) \leq \frac{\pi r^2}{G} , \quad (2)$$

is expected to hold for every concentric sphere of circumferential radius r . This, unfortunately, seems to fall short within the inner region of a black hole. As far as entropy packing is concerned, the interior of a black hole is apparently superfluous. The first local realization of maximal entropy packing has been demonstrated [4] within the context of spontaneously induced (G^{-1} treated as a VEV) general relativity. The black hole limit is then governed by a phase transition [5] which occurs precisely at the would have been horizon. The recovered exterior Schwarzschild solution of mass M connects with a novel core of vanishing spatial volume characterized by the non-singular Komar mass distribution

$$M(r) = \frac{r^2}{4G^2 M} . \quad (3)$$

The emerging onion-like maximal entropy packing profile is such that the entropy $S(r)$ of any inner sphere is unaffected by the outer layers; any additional incoming entropy is maximally packed on its own external layer. In some sense, each layer acts as a stretched horizon [6, 7]. Spontaneously induced general relativity sheds new light on the way information is stored within a black hole. Rather than envision bits of information evenly spread on the horizon surface, they now inhabit, universally and holographically, the entire black hole interior.

Within such a core, consider a light sheet interval [8] constructed as the difference between two concentric spherical cones (outer radius r_{out} , inner radius r_{in}) sharing a common solid angle $\Omega \leq 4\pi$. A light sheet *unit* interval is then defined to have order Planck area difference, that is

$$\Omega (r_{out}^2 - r_{in}^2) = \eta \ell_{Pl}^2 . \quad (4)$$

The $\mathcal{O}(1)$ coefficient η is regarded fundamental, and will be determined later. In this paper, rather than tiling the horizon by n Planck area patches, the traditional way, we suggest *filling up the interior with n light sheet unit intervals*. It is useful, but not mandatory, to think of a specific shape of these intervals; it is only their standard size and relative localization which are relevant. This way, we do not need to further assume several microstates per bit, but only count the total number g_n of distinguishable spatial arrangements of n indistinguishable degrees of freedom. At the fundamental level, the counting calls for graph enumeration which is currently unknown for large n . Consequently, recalling the discrete black hole prototype [9, 10], we construct a discrete holographic shell model, where the number of distinguishable configurations (tagged by a Wheeler's 'it from bit' binary code) is given by the Catalan series. While regarding ref.[4] to serve as the underlying rationale, our model may have life of its on. Discrete black holes have been conjectured [11] to be created in the lab, and the possibility of detecting terrestrial mini black holes has been recently raised [12].

By definition, a shell model assumes the arrangement of n degrees of freedom in concentric spherical shells, such that the i -th shell (counting from the origin outwards) hosts k_i degrees of freedom, with $\sum k_i = n$. Put in a different way, the exterior shell which hosts k degrees of freedom surrounds a core which exhibits a full shell structure for the residual $(n - k)$ degrees of freedom. All g_n different configurations are then being counted, without replication, by letting k run from 1 to n . Denoting by f_k the total number of distinguishable configurations of k degrees of freedom within a given shell, with $f_k < g_k$ (and $f_0 \leq g_0$) on consistency grounds, we are led to the

recursion formula

$$f_0 g_n = \sum_{k=1}^n f_k g_{n-k} , \quad (5)$$

which holds for $n = 1, 2, \dots$. Notice that, reflecting our universal approach, this formula does not necessarily depend on the total number of spacetime dimensions. The formal solution $g_n(f_1, \dots, f_n)$ of eq.5 is given by

$$\frac{g_n}{g_0} = \sum_{i,j} \frac{(j_1 + \dots + j_\ell)!}{j_1! \dots j_\ell!} \frac{f_{i_1}^{j_1} \dots f_{i_\ell}^{j_\ell}}{f_0^{j_1 + \dots + j_\ell}} , \quad (6)$$

where the positive integers i_p, j_p are subject to the Diophantine equation $\sum_p i_p j_p = n$. Obviously, there is no reason for the respective scales g_0 and f_0 to differ from each other, so we set $f_0 = g_0$ from this point on. Furthermore, we find it quite tempting, and presumably even required, to assign $f_0 = g_0 = 1$ for 'nothing'.

One must be more specific with regards to what defines a shell in the first place, at least at the algebraic level, by fixing or restricting the f_k series. Adopting the working principle that one and only one information storing mechanism exists, our discussion takes two paths.

A simple model: First, consider the possibility that the inner structure of any given shell is trivial. This triviality can be translated into

$$f_k = f_0 \implies g_n = 2^{n-1} g_0 , \quad (7)$$

giving rise to the statistical entropy

$$S_n = (n-1) \log 2 + \log g_0 \quad (n = 1, 2, \dots) . \quad (8)$$

The unfortunate feature that g_0 is not the zeroth element of the g_n series can be considered a drawback, telling us that the choice $g_0 = 1$ cannot be enforced in this case. In fact, such prototype discrete black hole models, either with $g_0 = 1$, or alternatively with $g_0 = 2$, have been discussed by Bekenstein and Mukhanov [10] and by many others [13], although not in the context of a shell model. A geometric representation of the simple shell structure is depicted in Fig.1.

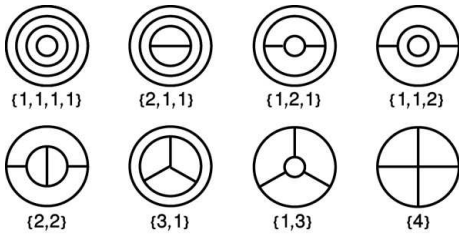


FIG. 1: A geometrical representation of the distinguishable configurations for the simple model is demonstrated for $n = 4$ (notice the equal length difference intervals). The shell structure of each configuration is specified.

An elaborate model: A non-trivial f_k must obey $f_k < g_k$, and is expected to exhibit a g_k -like structure. The simplest tenable choice is

$$f_k = g_{k-1} , \quad (9)$$

suggesting that it actually takes one degree of freedom to close a shell. In other words, resembling a Russian nested doll, each shell exhibits an inner shell structure, and so on. The emerging refined recursion formula, with eq.(9) substituted into eq.(5), is quite familiar and is known to generate the famous Catalan series; this series is given explicitly by $C_n = 1, 1, 2, 5, 14, 42, 132, \dots$. To be more specific, the total number of configurations is then

$$g_n = \frac{(2n)!}{(n+1)!n!} g_0 \quad (n = 0, 1, 2, \dots) , \quad (10)$$

and by means of Stirling's formula, the large- n entropy expansion takes the form

$$S_n \simeq n \log 4 - \frac{3}{2} \log n + \log \frac{g_0}{\sqrt{\pi}} + \dots . \quad (11)$$

Unlike in the simple model, this time g_0 does constitute the zeroth element of the g_n series. We may thus assign $g_0 = 1$ to the vacuum state to end up with $S_1 = 0$, and arrive at the non trivial yet far reaching conclusion that the associated black hole ground state is non-degenerate.

The emergence of the logarithmic term in the entropy expression eq.11, with the particular coefficient $-\frac{3}{2}$, is by no means trivial. With the Cardy formula [14] as a field theoretical starting point, first-order corrections to the Bekenstein-Hawking entropy have been calculated [15] for a variety of theories. Despite very different physical assumptions, these corrections are always proportional, with the same $-\frac{3}{2}$ factor, to the logarithm of the horizon surface area. The field theoretical corrections turn out to be in agreement with the Kaul-Majumdar [16] 'quantum geometrical' approach, and even with loop gravity models [17]. From any direction viewed, the universality of the $-\frac{3}{2}$ coefficient, shared by our elaborate model (eq.11), is manifest and still very mysterious.

There are many counting problems in combinatorics whose solution is given by the Catalan numbers, two of which are relevant to the present discussion. For example, C_n counts the number of expressions containing n pairs of parentheses which are correctly matched. In this language, $\{ \}$ is a fundamental building block, irreducible $\{ \dots \}$ stands for a single shell (with \dots denoting its inner shell structure), so that $\{ \dots \}, \dots, \{ \dots \}$ represents a full shell model configuration. Reading this configuration as a mathematical expression from left to right is equivalent to moving from the inside outwards. Adding a degree of freedom to a given configuration can be schematically described by closing from the r.h.s. by $\}$, and inserting $\{$ at some allowed inner location. Using the dictionary $\{ = 1$ and $\} = 0$, every distinguishable configuration is then tagged by a binary code which captures its

construction from individual indistinguishable degrees of freedom. This constitutes a realization of Wheeler's 'it from bit' philosophy; a missing ingredient, perhaps, is the rule which governs the transitions among the various configurations. A geometric representation of the distinguishable Catalan configurations (demonstrated for $n = 4$) is depicted in Fig.2 (notice that the equal area difference intervals are represented by equal length difference intervals). Evidently, the set of configurations of the simple model constitutes a subset of the Catalan configurations. Altogether, see Fig.3, there are exactly

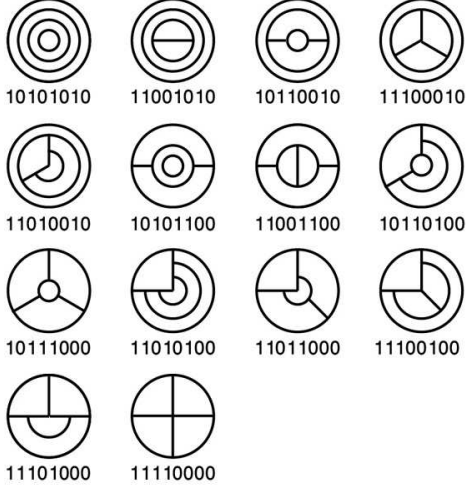


FIG. 2: A geometrical representation of the Catalan configurations is demonstrated for $n = 4$ (notice that the equal area difference intervals are represented by equal length difference intervals). Each configuration is tagged by a Wheeler style binary code.

C_n ordered roots to construct an order- n black hole by adding, step by step, fundamental order-1 black holes.

In computer science, a data structure is a particular way of storing and organizing data so that it can be used efficiently. The Catalan numbers happen to be directly related to a well known restricted data structure called a 'stack', thereby suggesting that even Nature's ultimate information storage, a black hole, exhibits a text book standard data structure. A stack is nothing but a last-in-first-out (LIFO) data structure whose limited number of operations solely consists of 'push' (\equiv open parentheses) and 'pop' (\equiv close parentheses). C_n is then the number of stack-sortable permutations of the ordered list of integers $1, 2, \dots, n$. Beware that the analogy is solely in the count procedure, and information does not really exit the black hole the way it does from a standard data structure.

A direct consequence of filling up the entire black hole interior by an arbitrary integer number of light sheet unit intervals (define by eq.4) is the area quantization

$$A_n = nA_1. \quad (12)$$

This is fully consistent with the apparent role of the hori-

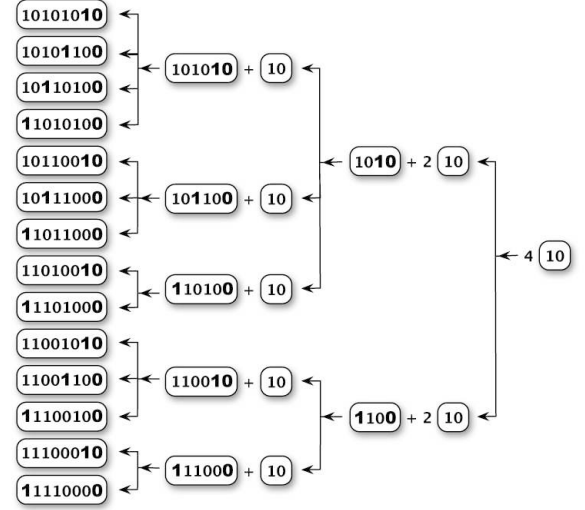


FIG. 3: The construction tree of an order- n black hole from n fundamental black holes is sketched for $n = 4$. The 1, 0 digits associated with the last-in-first-out degree of freedom are highlighted.

zon surface area as a quantum mechanical adiabatic invariant [9], where an optional constant term has been dropped off to assure $A_0 = 0$. The consistency of eq.12 is also reflected by the fact that by its substitution into the modified (logarithm corrections included) asymptotic area entropy formula derived by a variety of authors [14–17], eq.11 gets forcefully recovered.

Only our simple model is capable of literally reconciling with the Bekenstein-Hawking black hole entropy formula. Eq.8 and eq.12 can coincide, with

$$A_1 = 4\ell_{Pl}^2 \log 2, \quad S_1 = \log 2, \quad M_1 = \sqrt{\frac{\log 2}{\pi}} M_{Pl}, \quad (13)$$

provided one is ready to accept a doubly degenerate ground state, that is $g_1 = 2$. The non-degenerate case $g_1 = 1$, for which $S_1 = 0$, may also fit in, but it requires supplementing the Bekenstein-Hawking area entropy by a constant term, which is of course legitimate.

New doors get open, however, in case the Bekenstein-Hawking entropy formula happens to be, albeit mandatory, just a large- n limit. In particular, as far as our elaborate model is concerned, we do asymptotically recover the equal spacing feature $\Delta S = 2 \log 2$ up to $\mathcal{O}(\frac{1}{n})$ corrections. This allows us to calculate the A_1 area coefficient, so that the fundamental ground state black hole is now associated with

$$A_1 = 8\ell_{Pl}^2 \log 2, \quad S_1 = 0, \quad M_1 = \sqrt{\frac{\log 2}{2\pi}} M_{Pl}, \quad (14)$$

and thereby determine the fundamental coefficient η , see eq.4, to be $8 \log 2$. Notice that the area is now twice as large in comparison with the simple model.

The mass distribution which accompanies the holographic shell model is in some respect counter intuitive. It can be easily verified that the overall mass of an order- n black hole is given by

$$M_n = \frac{1}{4G} \sqrt{\frac{nA_1}{\pi}}. \quad (15)$$

But what about the partial mass $M_{k,n}$ associated with the order- k inner concentric section which resides inside a larger order- n black hole? Recalling that $S_{k,n} = S_k$, a fact which expresses the universal nature of the holographic entropy packing, one could have naively expected $M_{k,n} = M_k$, but this is certainly not the case. A closer inspection, based on eq.3, reveals that

$$M_{k,n} = \frac{k}{n} M_n = \sqrt{\frac{k}{n}} M_k < M_k, \quad (16)$$

$$\frac{S_{k,n}}{M_{k,n}} = \sqrt{\frac{n}{k}} \frac{S_k}{M_k} > \frac{S_k}{M_k}. \quad (17)$$

The conclusion is threefold:

- (i) The mass per degree of freedom, namely M_n/n , depends on the black hole size. It costs much less gravitational energy to store one degree of freedom inside a large black hole than inside a small black hole.
- (ii) For a given n , however, the allocation of mass per degree of freedom, that is $M_{k,n}/k$, is k -independent. All degrees of freedom are worth the same, a true equipartition, no matter where they are located. This pleasing result accounts for the self consistency of the shell model.
- (iii) Reflecting the $\sqrt{\frac{n}{k}} > 1$ factor, the entropy to energy ratio calculated for the order- k inner core section is in apparent violation of Bekenstein's universal entropy bound [18]. It is thus important to note that Bekenstein's bound is relevant only for weakly self gravitating isolated physical systems, which is not the case here, and for these it is a much stronger bound than the holographic one.

To summarize, the discrete holographic shell model has been carefully designed to capture the holographic entropy packing inside black holes. In this model, the light sheet unit interval has been elevated to the level of the fundamental geometrical building block. This way, we conceptually deviate from the traditional dogma of tiling the black hole horizon by Planck area patches. By counting the total number of distinguishable configurations, and tagging them with a binary code, the Catalan series has made a remarkable entrance into black hole physics. The area entropy formula has been recovered, including in particular its universal logarithmic correction, and the equipartition of mass per degree of freedom proven. The conclusion that Nature's ultimate information storage resembles a text book standard data structure sheds new light on the black hole information puzzle.

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